Symmetry restoration in the early Universe

J. Phys. A: Math. Gen., Vol. 11, No. 7, 1978. Printed in Great Britain

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Received 6 February 1978

Abstract. I discuss a simple model in which the vacuum expectation value of a complex scalar field ϕ is non-zero in the flat-space vacuum. Generalising to curved space and choosing the conformal vacuum in Friedmann models I find that in hot big bang models symmetry is restored at early times whereas in cold models two new effects arise—a classical curvature term and a zero-point fluctuation term. These have opposite effects and lead to restoration only in closed universes.

1. The model

This is a straightforward adaption of that discussed by Kirzhnits and Linde (1976). The Lagrangian is

$$L = -\tfrac{1}{2}(|\nabla\phi|^2 - \mu^2|\phi|^2 + \tfrac{1}{6}R|\phi|^2 + \lambda|\phi|^4) \tag{1}$$

R is the Ricci scalar. L is chosen so that the well known scaling properties of $\lambda\phi^4$ theory in flat space are generalised to curved space. The resulting Euler–Lagrange equation

$$(-\nabla_\alpha\nabla^\alpha + \tfrac{1}{6}R - \mu^2 + 2\lambda|\phi|^2)\phi = 0 \tag{2}$$

has the property that if $\{\phi, g_{\alpha\beta}\}$ solve (2) then $\tilde{\phi} = \Omega^{-1}\phi$ and $\tilde{g}_{\alpha\beta} = \Omega^2 g_{\alpha\beta}$ are such that

$$(-\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha + \tfrac{1}{6}\tilde{R} - \mu^2\Omega^{-2} + 2\lambda|\tilde{\phi}|^2)\tilde{\phi} = 0 \tag{3}$$

where $\tilde{\nabla}_\alpha$ and $\tilde{R}$ denote the covariant derivative and Ricci scalar of the rescaled metric $\tilde{g}_{\alpha\beta}$. If $\mu^2 = 0$ the equation is conformally (or Weyl) invariant.

In flat space the 'vacuum' of the model has a non-vanishing expectation value for ϕ. Following Kirzhnits and Linde (1976) I write

$$\phi = \frac{1}{\sqrt{2}}(\sigma + \phi_1 + \mathrm{i}\phi_2). \tag{4}$$

To lowest order, averaging the equation of motion in the Gibbs state at temperature T, I obtain an equation for σ:

$$\sigma[\mu^2 - \lambda(\sigma^2 + \Phi)] \tag{5}$$

$$\Phi = 3\langle\phi_1^2\rangle + \langle\phi_2^2\rangle. \tag{6}$$

Φ represents the thermal fluctuations and the angular brackets represent averaging in the Gibbs state. The following two self-consistent solutions are possible.

0305-4770/78/0007-1341$01.00

1.1. *Disordered state*

$\sigma = 0$, ϕ_1 and ϕ_2 have masses

$$m_1^2 = m_2^2 = -\mu^2 + \lambda\Phi.$$

1.2. *Ordered state*

$$\sigma = \frac{\mu^2}{\lambda} - \Phi$$

$$m_1^2 = -\mu^2 + \lambda(3\sigma^2 + \Phi)$$

$$m_2^2 = 0.$$

The massless particle is of course the Goldstone boson. Obviously as Φ increases the disordered state is favoured and above a certain value no ordered solution is possible. If m_1^2 and m_2^2 are approximated by zero the mean square fluctuation of a real field ϕ_1 is, according to Kirzhnits and Linde,

$$\langle\phi^2\rangle = T^2/12 \tag{7}$$

so that the ordered state is impossible above a certain temperature

$$T_c = \mu\left(\frac{3}{\lambda}\right)^{1/2}.$$

2. Generalisation to Friedmann universe

Any Friedmann model has the form

$$ds^2 = \Omega^2(\eta)(-d\eta^2 + g\,d\Omega_k^2) \tag{8}$$

where $\Omega(\eta)$ is the scale factor and $d\Omega_k^2$ is the metric of a three-space of constant curvature with radius a. The conformally rescaled field equation becomes

$$\left(\frac{\partial^2}{\partial\eta^2} - \nabla_k^2 + \frac{k}{a^2} - \mu^2\Omega^2(\eta) + 2|\tilde{\phi}|^2\right)\tilde{\phi} = 0 \tag{9}$$

where ∇_k^2 is the Laplacian on the three space. I now treat this equation in the same way as in flat space but choosing the Gibbs state in the rescaled, static space. The rescaled temperature remains constant (Gibbons and Perry 1977). We also ignore the η dependence of Ω. One would expect this 'adiabatic' approximation to be valid provided the Hubble time $\Omega/(\partial\Omega/\partial\eta)$ is long compared with μ^{-1} which will be true in the regions of interest. The result is an equation similar to (5) but with an extra classical term:

$$\sigma\left(\mu^2\Omega^2(\eta) - \frac{k}{a^2} - \lambda(\sigma^2 + \Phi)\right) = 0. \tag{10}$$

Φ contains the fluctuations. This term is formally divergent and must be regularised. I propose using zeta function regularisation (Dowker and Critchley 1976, Hawking 1977).

As explained in (Gibbons and Perry 1977) we may compute quantities in the Gibbs state working on Riemannian space obtained by Wick rotating the time coordinate and imposing a periodicity $\beta = T^{-1}$. For a real field of mass m the partition function is given by

$$\ln Z = \tfrac{1}{2}[\zeta'(0) + \ln(2\pi\mu_r^2)\zeta(0)] \tag{11}$$

where

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\, e^{-m^2 t} Y(t)\, dt \tag{12}$$

$$Y(t) = \sum e^{-\lambda_n t} \tag{13}$$

λ_n are the eigenvalues of the zero-mass elliptic operator governing the fluctuations and μ_r is the 'renormalisation mass'. The mean fluctuations are given by

$$\langle\phi^2\rangle = -2\frac{\partial \ln Z}{\partial \eta^2}\left(\int \sqrt{g}\, d^4x\right)^{-1}. \tag{14}$$

This yields, in the limit $m^2 \to 0$:

$$\langle\phi^2\rangle \int \sqrt{g}\, d^4x = \frac{\partial}{\partial s} s\zeta(s+1)|_{s=0} + \ln(2\pi\mu_r^2) s\zeta(s+1)|_{s=0} \tag{15}$$

if $s\zeta(s+1)|_{s=0} = 0$ the answer is independent of the renormalisation mass.

General theory shows that

$$\zeta(1+\epsilon) = \frac{1}{\epsilon}\int \frac{B_1\sqrt{g}\, d^4x}{(4\pi)^2} + O(1) \tag{16}$$

for small ϵ. B_1 is the relevant coefficient in the Hadamard–Minakshisundarem–de Witt (='Hamidew) expansion. For the massless, conformally invariant equation $B_1 = 0$ and the fluctuations reduce to

$$\langle\phi^2\rangle = \zeta(1)\left(\int \sqrt{g}\, d^4x\right)^{-1} \tag{17}$$

which is independent of the renormalisation mass. Using the results of Gibbons (1977) I obtain

$$\langle\phi^2\rangle \int \sqrt{g}\, d^4x = \zeta_H(\tfrac{1}{2}) \tag{18}$$

at low temperatures and

$$\langle\phi^2\rangle = T^2/12 \tag{19}$$

at high temperatures, in agreement with Kirzhnits and Linde. $\zeta_H(s)$ is the Hamiltonian zeta function defined by

$$\zeta_H(s) = \sum \frac{1}{E_n^{2s}} \tag{20}$$

E_n being the energy eigenvalues of the field. Again using the results of Gibbons (1977) I obtain

$$\langle\phi^2\rangle = -k/48a^2. \tag{21}$$

This latter result agrees with unpublished work of J S Dowker who uses a different method. He subtracts off the direct part of a Green function. Strictly speaking the zeta function method as I have defined it produces (21) only for $k>0$ but I shall assume it true for $k<0$. Dowker's results appear to support this. It is interesting that the fluctuations can be *negative* which is formally impossible and a result of the regularisation scheme which violates naive formal inequalities. One might worry that this invalidates the interpretation of $\langle\phi^2\rangle$ as fluctuations, nevertheless in this paper I shall proceed as in the flat-space case.

Let us suppose that we are in the zero-temperature state (cold big bang). We find that σ satisfies:

$$\sigma\left[\mu^2\Omega^2(\eta)-\frac{k}{a^2}-\lambda\left(\sigma^2-\frac{k}{12a^2}\right)\right]=0. \tag{22}$$

Thus the ordered phase satisfies

$$\sigma^2=\frac{\mu^2\Omega^2(\eta)}{\lambda}-\frac{k}{a^2}\left(1-\frac{\lambda}{12}\right)\frac{1}{\lambda}. \tag{23}$$

As $\eta\to 0$, $\Omega\to 0$. This means that at early times the ordered phase will only be possible if $k(1-\frac{1}{12}\lambda)>0$. Since one usually has $\lambda\ll 1$ this depends on the sign of k. In closed models an ordered phase at early times is not possible whereas for an open model it will be favoured. As stated in the abstract the classical term and the zero-point fluctuations have the opposite effect.

On the other hand if one considers a more realistic example in which the system is at a finite temperature 3 K the thermal fluctuations will swamp both the classical term and the zero-point term and at high enough temperatures the disordered state is favoured. As the universe cools down it will 'fix' in the ordered state. This has been discussed by Kirzhnits and Linde.

Another application of these ideas would be to the de Sitter invariant state on de Sitter's space. The eigenvalues of $-\nabla_\alpha\nabla^\alpha+\frac{1}{6}R$ are

$$\tfrac{1}{3}\Lambda[n(n+3)+2] \tag{24}$$

and the degeneracies are

$$\tfrac{1}{6}(n+1)(n+2)(2n+3). \tag{25}$$

This leads directly to

$$\zeta_c(s)=\left(\frac{12}{\Lambda}\right)^s\frac{1}{24}\sum_{n=3,5,\ldots}\frac{n}{(n^2-1)^{s-1}} \tag{26}$$

$$=\left(\frac{12}{\Lambda}\right)^s\frac{1}{24}\sum_{r=0}^{\infty}\frac{1}{r!}\frac{\Gamma(s+r-1)}{\Gamma(s-1)}\left[\left(1-\frac{1}{2^{2s+2r-3}}\right)\zeta(2s+2r-3)-1^{2s}\right] \tag{27}$$

whence

$$\zeta_c(1)=-1/3\Lambda \tag{28}$$

$$\langle\phi^2\rangle=-\Lambda/72\pi^2 \tag{29}$$

which also agrees with the unpublished work of J S Dowker. The fact that $\langle\phi^2\rangle$ is negative is especially interesting because this state has an interpretation as a Gibbs state of temperature $T=(2\pi)^{-1}\sqrt{(\frac{1}{3}\lambda)}$. Nevertheless the fluctuations are *negative.*

Now the corresponding equation for σ is

$$\sigma = \left(\mu^2 - \frac{R}{6}\right)\frac{1}{\lambda} - \Phi = \left(\mu^2 - \frac{2\Lambda}{3}\right)\frac{1}{\lambda} - \frac{1}{18\pi^2}\Lambda. \tag{30}$$

If $\lambda \ll 1$ it is the classical term which governs whether an ordered or disordered state appears. However usually $\mu^2 \gg \Lambda$ and geometrical effects are negligible.

One may also repeat the analysis for $\mathbb{CP}2$ (Gibbons and Pope 1978) and find that

$$\zeta_c(s) = \left(\frac{3}{2\Lambda}\right)^s \zeta(2s-3) \tag{31}$$

$$\langle\phi^2\rangle = -\frac{\Lambda}{144\pi^2}. \tag{32}$$

Again the fluctuations are negative.

Acknowledgments

This work was begun at the Max Planck Institüt in Munich and substantially helped by discussions with J S Dowker at the Vancouver workshop on Quantum Gravity organised by W Unruh in August 1977. I should like to thank Bill Unruh for his hospitality and Stuart Dowker for sharing his knowledge.

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